Elementi di probabilitá e Statistica

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Outline

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Elementary Probability Theory

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- Sample Space
- Probability Measures
- Independence
- Conditional Probabilities
- Bayesian Inversion
- Partitions

Sample Space

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The sample space is a set of elementary outcomes. An event is a subset of the sample space. Sample spaces are often denoted by Ω and events are often called $A, B, C, \ldots$.

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Example

Dado. $\Omega = \{'1', '2', \ldots, '6'\}$

- Un tiro del dado in cui si ottiene '1' da' luogo all'evento $\{'1'\}$:
- "Il risultato é meno di 4" consiste nell'evento: $\{'1', '2', '3'\}$
- il numero totale di eventi coincide con il numero totale di sottoinsiemi di Ω.
- Nota: $'1' \neq \{'1'\}$

Probability Measures

Una funzione P a valori reali sullo spazio degli eventi 2^Ω e' una funzione di probabilitá **sse**:

Axioms

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Esempio di Ω: "Il risultato di un tiro di dado e' minore di 7".

Probability Measures

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Consequences

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- $P(\emptyset) = 0$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Probability Measures

La situazione in cui due eventi A e B occorrono insieme ammette una probabilita' pari a $P(A \cap B)$.
La conoscenza di un evento B cambia la nostra aspettativa (e quindi la probabilita') di un secondo evento A. Quando questo non avviene allora i due eventi si dicono *indipendenti*.

Independence

A is independent from $B \iff P(A \cap B) = P(A) \cdot P(B)$

Probability Measures

Conditional Probabilities

The probability of A given an event B is written as $P(A|B)$ and it is given by:

$P(A|B) = \frac{P(A \cap B)}{P(B)}$

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Note that:

- $P(A|A) = 1$, $P(A|\bar{A}) = 0$
- If A and B are independent: $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A) \cdot P(B)}{P(B)} = P(A)$

Bayesian Inversion

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In the Bayes formula

$$\boxed{P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}}$$

the posteriors $P(B|A)$ are used instead of $P(A|B)$.

Partitions

When a partion in n events $A_i \quad (i = 1, \ldots, n)$ is available for Ω, i.e.

$$\begin{cases} \quad \Omega & = & \bigcup_{i=1}^{n} A_i \\ \forall i \neq j \quad A_i \cap A_j & = & \emptyset \end{cases}$$

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$$= \sum_{i}^{n} P(B \cap A_i) = \sum_{i}^{n} P(B|A_i) P(A_i)$$

Stochastic Variables

Stochastic Variables

- Distribution Functions
- Probability Measures
- Discrete and Continuous Stochastic Variables
- Frequency Function
- Expectation Value
- Variance
- Two dimensional Stochastic Variables

Stochastic Variable

Sample space of a stochastic variable

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Thus if $u \in \Omega$ then $\xi(u) \in R$.

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In the "Dado" example, the image of ξ is $\{1,...,6\}$, and $\xi('1') = 1,...,\xi('6') = 6$.

Stochastic Variables

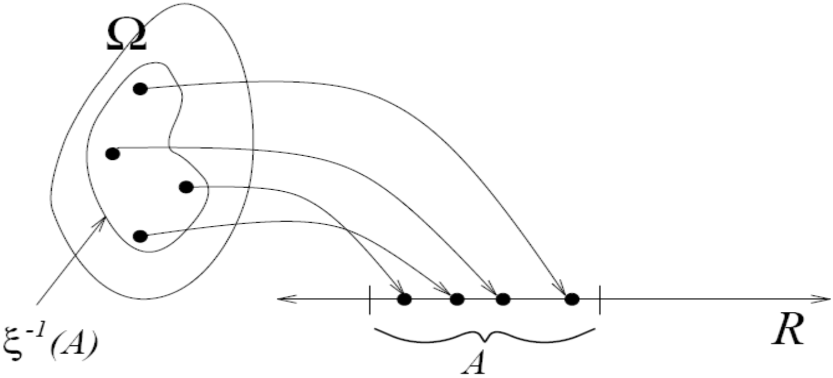

Figure 1.3: $P(\xi \in A)$ is defined through $P(\xi^{-1}(A)) = P(\{u : \xi(u) \in A\})$.

Stochastic Variables

Sample space of a stochastic variable

The image of the sample space Ω in R under the random variable ξ, i.e. the range of ξ, is called *the sample space of the stochastic variable* ξ and is denoted by Ω_ξ.
In short, $\Omega_\xi = \xi(\Omega)$

Distribution Function

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Let A be a subset of R and consider the inverse image of A under ξ, i.e. $\xi^{-1}(A) = \{u \in \Omega : \xi(u) \in A\} \subseteq \Omega$.

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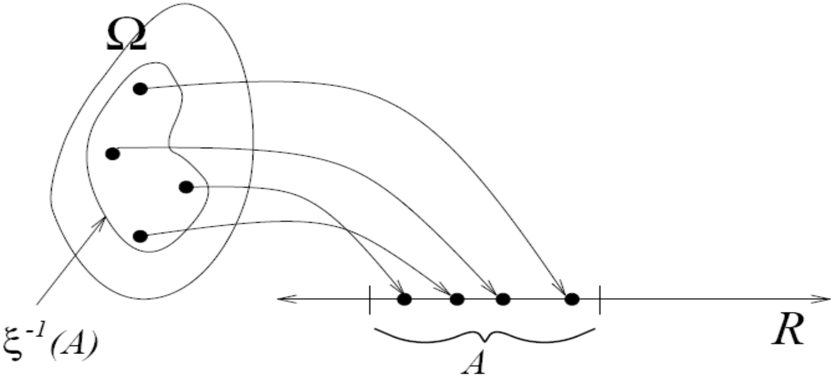
If A is the interval $(-\infty, x]$ then the real-valued function F denoted by
$F(x) = P(\{u : \xi(u) \leq x\}) = P(\xi \leq x) \qquad \forall x \in R$
is called the *distribution function of the random variable* ξ.
Sometimes F is denoted F_ξ to indicate that it is the distribution function of the particular random variable ξ.

Stochastic Variables



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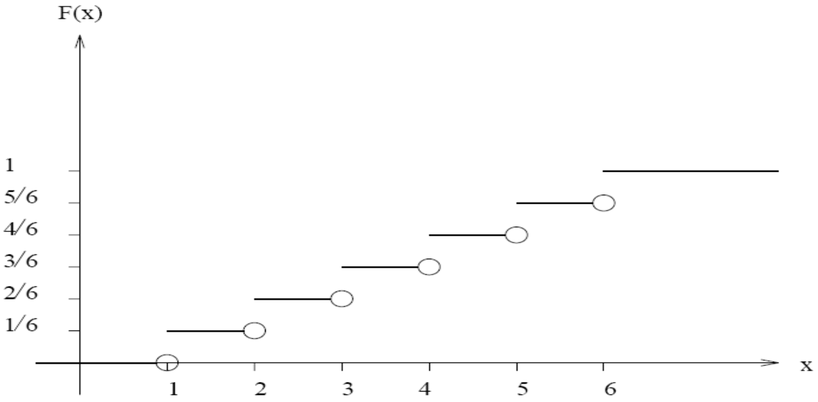


Figure 1.4: Fair die: Graph of the distribution function.

Frequency Function

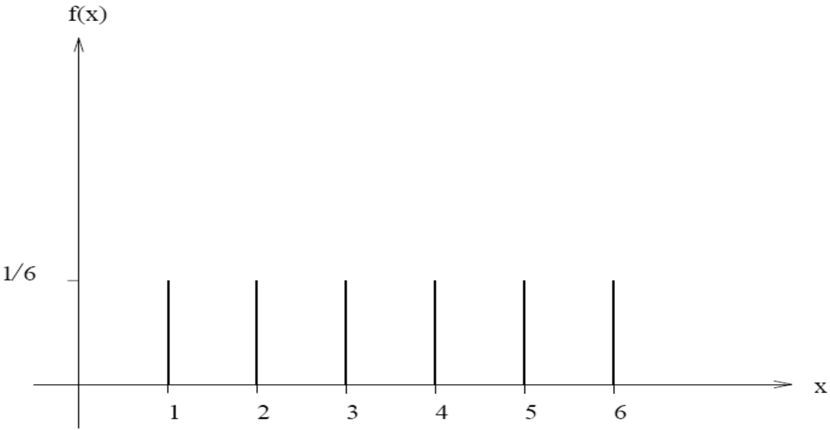
Frequency Function

Another way of seeing the distribution of a random variable is through its *frequency function*, f, given by:

- Discrete Case: $f(x) = P(\xi = x)$
- Continuous Case: $f(x) = F'(x) = \frac{dF(x)}{dx}$

In order to explicit the reference to the random varable ξ f is often denoted as f_ξ.

Frequency Function



Frequency Function and Probabilities

Frequency and Distribution Function

The probability distribution of a random variable ξ can be computed from its frequency function f_ξ as follows:

- Discrete Case: $P(\xi \in A) = \sum_{x \in A} f_\xi(x)$
- Continuous Case: $P(\xi \in A) = \int_A f_\xi(x)dx$

Frequency Function and Probabilities

Consequences

- Discrete Case:

$$P(\Omega_\xi) = \sum_{x \in \Omega_\xi} f_\xi(x) = 1$$

- Continuous Case:

$$P(\Omega_\xi) = \int_{-\infty}^{+\infty} f_\xi(x)dx = 1$$

Expectation

Expectation or Mean value

A way to summaize the distribution of a random variable is through its *expectation value*, or statistical mean, $E[\xi]$, given by:

- Discrete Case:

$$E[\xi] = \sum_{x \in \Omega_\xi} x \cdot f_\xi(x) = \sum_i x_i \cdot f_\xi(x_i)$$

- Continuous Case:

$$E[\xi] = \int_{-\infty}^{+\infty} x \cdot f_\xi(x) dx$$

In both cases $E[\xi]$ is often denoted by μ.

Variance

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A second aspect is to express how much is the mean value of a random variable is representative of the entire distribution. This is given by the notion of standard deviation or, more commonly, the *variance* $Var[\xi]$:

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- Discrete Case:

$$Var[\xi] = \sum_{x \in \Omega_\xi} (x-\mu)^2 \cdot f_\xi(x) = \sum_i (x_i - \mu)^2 \cdot f_\xi(x_i)$$

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The variance of a variable ξ is often denoted by σ^2, whereas σ denotes the *standard deviation*.

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In the "Dado" example obviously follows:

- $E[\xi] = \sum_{i=1}^{6} \frac{1}{6} \cdot i = \frac{6 \cdot (6+1)}{2} \cdot \frac{1}{6} = \frac{7}{2}$
- $Var[\xi] = \sum_{i=1}^{6} (i - \frac{7}{2})^2 \cdot \frac{1}{6} = \frac{35}{12}$

Frequency Function

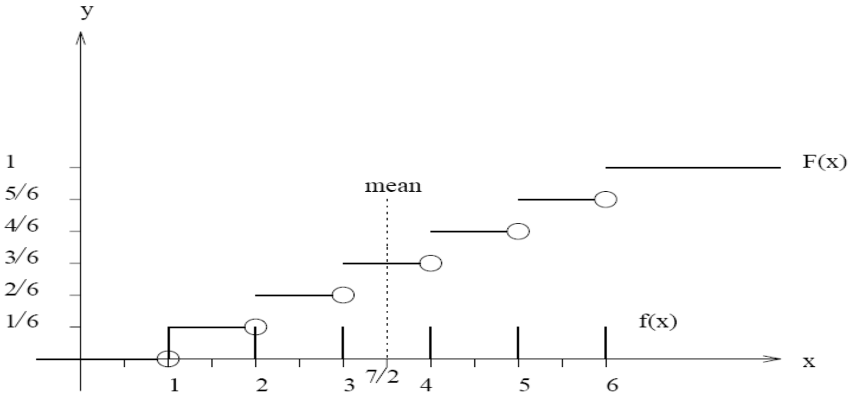

Figure 1.6: Fair die: Expectation value (mean)

Multiple Random Variables

Multiple Variable

Let ξ and η be two random variables defined on the same sample space Ω.

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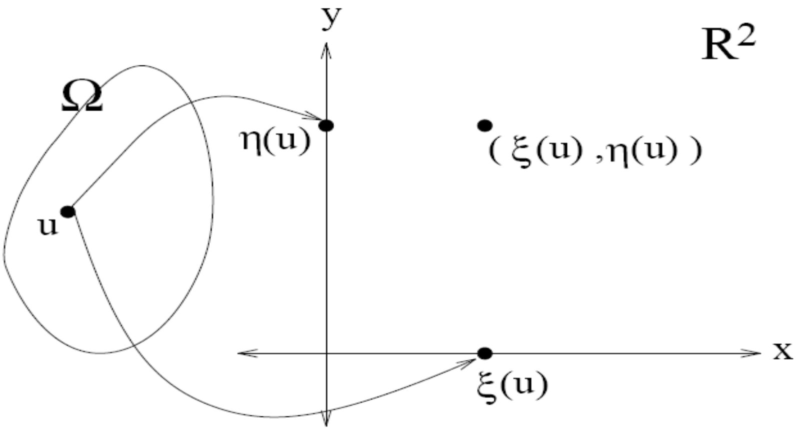
Then (ξ,η) is a two-dimensional random variable from Ω to $\Omega_{(\xi,\eta)} = \{(\xi(u),\eta(u)) : u \in \Omega\} \subseteq R^2$.

Here $R^2 = R \times R$ is the Cartesian product of the set of real numbers R with itself.

Multiple Random Variables



Multiple Random Variables

Generalizations: Discrete Case

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The frequency function f of (ξ, η) is then defined by:

$$f(x,y) = P(\xi = x, \eta = y) = P((\xi, \eta) = (x,y)) \qquad \forall (x,y) \in R^2$$

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Furthermore:

$\forall A \subseteq \Omega_{(\xi,\eta)}$

$$P(A) = P((\xi, \eta) \in A) = \sum_{(x,y) \in A} f(x,y)$$

Multiple Random Variables

Marginal distributions

We can recover the frequency functions of either of the individual variables by summing or integrating over the other.

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$$f_\xi(x) = \sum_{y \in \Omega_\eta} f(x, y)$$

$$f_\eta(y) = \sum_{x \in \Omega_\xi} f(x, y)$$

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$$f_\eta(y) = \sum_{x \in \Omega_\xi} f(x, y)$$

In this context f_ξ and f_η are often referred to as the marginal distributions of ξ and η respectively.

Functions over Multiple Random Variables

Special functions $\Psi(u)$ of two random variables (i.e. $\Psi(u) = g(\xi(u), \eta(u))$ can be easily derived from the single variable case.

Mean

The *expectation value* of $g(\xi, \eta)$ when (ξ, η) is discrete, is given by:

$$E[g(\xi, \eta)] = \sum_{(x,y) \in \Omega_{(\xi,\eta)}} g(x,y) \cdot f_{(\xi,\eta)}(x,y).$$

Expectation (Continuous Case)

$E[g(\xi, \eta)] = \int_{-\infty}^{+\infty} g(x,y) \cdot f_{(\xi,\eta)}(x,y) dx\, dy$ if (ξ, η) is continuous.

Stochastic or Random Processes

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Independence

Note that the random variables are in general not independent (i.e. $P(\xi_{t+1}|\xi_t) \neq P(\xi_{t+1})$ in general). In fact, the interesting thing about stochastic processes is the dependence between the random variables ξ_{t+1} and ξ_t, for the different t.

Selected Probability Distributions

Useful Distribution

- Binomial Distribution
- Normal Distribution
- Other Distributions
- Distribution Tables
- Probability Measures

See them in (Krenn & Samuelsson, 1997)

References

Introduction to Probability

- (Krenn & Samuelsson, 1997), Brigitte Krenn, Christer Samuelsson, *The Linguist's Guide to Statistics Don't Panic*, Univ. of Saarlandes, 1997.
  URL:
  `http://nlp.stanford.edu/fsnlp/dontpanic.pdf`