Elementi di Teoria dell'Informazione

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Outline

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- Information Theory
- Entropy
- Joint-Entropy and Conditional entropy
- Mutual Information
- Cross-Entropy and Norms

Information Theory

Let ξ be a discrete stochastic variable with a finite range $\Omega_\xi = \{x_1, \ldots, x_M\}$ and let $p_i = p(x_i)$ be the corresponding probabilities.

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Or equivalently:

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This is the information needed to specify which of the x_i has occurred. The problem is writing ξ.
Let us assume further that we only have a small set of symbols $A = \{a_k : k = 1, \ldots D\}$, that is a *coding alphabet*.

Information Theory

Thus each x_i will be represented by a string over A.
Let us assume that ξ is *uniformly distributed*, i.e.

$$p_i = \frac{1}{M} \qquad \forall i = 1, \ldots, M,$$

and that the coding alphabet is exactly $A = \{0, 1\}$.

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Thus, each x_i will be represented by a binary number. To use N binary digits to specify which x_i actually occurred means:

$$N : 2^{N-1} < M \leq 2^N$$

Thus we need $N = \lceil \log_2 M \rceil$ digits.
So what if the distribution is *nonuniform*, i.e., if the p_is are not all equal?

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Example: if $p_i \approx 1$ then $M_{p_i} \approx 1$.

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For a binary coding alphabet, we thus need

$$\log_2 M_{p_i} = \log_2 \frac{1}{p_i} = -\log_2 p_i$$

binary digits to specify that the outcome was x_i.
Thus, the uncertainty introduced by p_i is in the general case

$$-\log_2 p_i$$

Entropy

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The uncertainty introduced by the random variable ξ will be taken to be the *expectation value of the number of digits required to specify its outcome*.
This is the expectation value of $-\log_2 P(\xi)$, i.e.

$$E[-\log_2 P(\xi)] = \sum_i -p_i \log_2 p_i$$

Entropy

Entropy

The entropy $H[\xi]$ of ξ is precisely the amount of uncertainty introduced by the random variable ξ and it is more often referred to a natural logarithm $\ln(.)$, so that

$$H[\xi] = E[-\ln p(\xi)] = \sum_{x_i \in \Omega_\xi} -p(x_i) \ln p(x_i) = \sum_i^M -p_i \ln p_i$$

Entropy

Example 1: Dado

In the Dado example, $\forall i = 1, \ldots, 6$, it follows that $p_i = \frac{1}{6}$.

$$H[\xi] = E[-\ln p(\xi)] = \sum_{x_i \in \Omega_\xi} -p(x_i) \ln p(x_i) = 6 \cdot \frac{1}{6} \ln 6 = 1,792$$

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Example 2: Dado Perdente

A loosing Die: $p_1 = 1.00$, and $\forall i = 2, \ldots, 6$, $p_i = 0$.

$$H[\xi] = E[-\ln p(\xi)] = \sum_{x_i \in \Omega_\xi} -p(x_i) \ln p(x_i) = 1 \ln 1 = 0$$

Entropy

Consequence

Given a distribution $p_i \quad (i = 1, , \ldots, M)$ for a discrete random variable ξ then for any other distribution $q_i \quad (i = 1, , \ldots, M)$ over the same sample space Ω_ξ it follows that:

$$H[\xi] = -\sum_i^M p_i \ln p_i \leq -\sum_i^M p_i \ln q_i$$

where equality holds **iff** the two distribution are the same, i.e. $\forall i = 1, \ldots, M \qquad p_i = q_i$

Joint-Entropy

Given two random variable ξ and η:

Joint-Entropy

the *joint entropy* of ξ and η is defined as:

$$H[\xi,\eta] = -\sum_{i=1}^{M}\sum_{j=1}^{L} p(x_i,y_j)\ln p(x_i,y_j)$$

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Conditional-entropy

Conditional Entropy

the *conditional entropy* $H[\xi|\eta]$ of ξ and η is defined as:

$$\begin{aligned} H[\xi|\eta] &= -\sum_{j=1}^{L} p(y_j) \sum_{i=1}^{M} p(x_i|y_j) \ln p(x_i|y_j) = \\ &= -\sum_{j=1}^{L} \sum_{i=1}^{M} p(x_i, y_j) \ln p(x_i|y_j) \end{aligned}$$

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Conveyed Information

The *information conveyed* by η, denoted $I[\xi|\eta]$, is the reduction in entropy of ξ by finding out the outcome of η. This is defined by:

$$I[\xi|\eta] = H[\xi] - H[\xi|\eta]$$

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Consequences

Note that:

$$\begin{aligned}
I[\xi|\eta] &= H[\xi] - H[\xi|\eta] = H[\xi] - (H[\xi,\eta] - H[\eta]) = \\
&= H[\xi] + H[\eta] - H[\xi,\eta] = H[\xi] + H[\eta] - H[\eta,\xi] = \\
&= H[\eta] + H[\xi] - H[\eta,\xi] = H[\eta] - H[\eta|\xi] = \\
&= I[\eta|\xi]
\end{aligned}$$

Mutual Information

Given two random variable ξ and η:

Mutual Information

the *mutual information* between ξ and η is defined as:

$$\begin{aligned}
MI[\xi,\eta] &= E[\ln \frac{P(\xi,\eta)}{P(\xi)\cdot P(\eta)}] = \\
&= \sum_{(x,y)\in\Omega_{(\xi,\eta)}} f_{(\xi,\eta)}(x,y) \ln \frac{f_{(\xi,\eta)}(x,y)}{f_\xi(x)\cdot f_\eta(y)}
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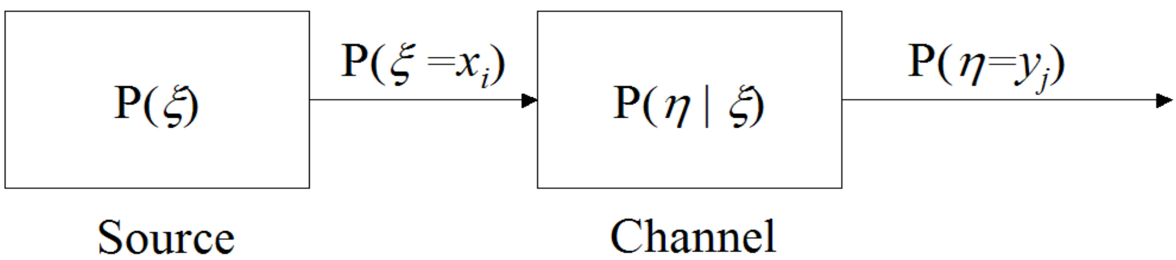

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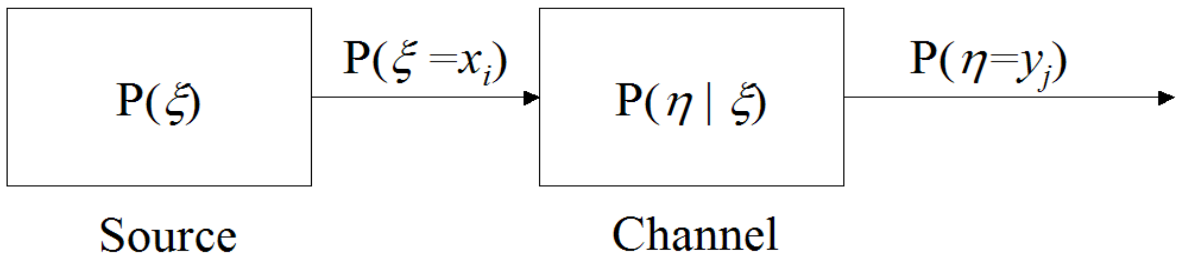

How much information about the source output x_i does an observer gain by knowing the channel output y_j?

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Mutual Information measures the amount of information about a random variable ξ an observer receives when the outcome of a random variable η is known, in fact:

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$$\begin{aligned} MI[\xi,\eta] &= H[\xi] - H[\xi|\eta] = \\ &= \sum_{(x,y)\in\Omega_{(\xi,\eta)}} f_{(\xi,\eta)}(x,y) \ln \frac{f_{(\xi,\eta)}(x,y)}{f_\xi(x)\cdot f_\eta(y)} \end{aligned}$$

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$H[\xi,\eta] = H[\eta,\xi]$
$H[\xi,\eta] = H[\eta] + H[\xi|\eta], \qquad H[\xi|\eta] = H[\xi,\eta] - H[\eta]$

Symmetry

Note that mutual information is symmetric in ξ and η, that is $MI[\xi,\eta] = MI[\eta,\xi]$, as

$$H[\xi] - H[\xi|\eta] = H[\xi] + H[\eta] - H[\xi,\eta] = H[\eta] - H[\eta|\xi]$$

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Given the two random variable ξ and η: the *pointwise mutual information* between $\xi = x_i$ and $\eta = y_j$ is defined as:

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Use of the pmi

If $MI[x_i, y_j] >> 0$, there is a strong correlation between x_i and y_j
If $MI[x_i, y_j] << 0$, there is a strong negative correlation.
When $MI[x_i, y_j] \approx 0$ the two outcomes are almost independent.

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Example

Predicting the next w of a sequence of n words $w_k \in Dict$:

$$P(\xi_n = w | \xi_{n-1} = w_{n-1}, \xi_{n-2} = w_{n-2}, \ldots, \xi_1 = w_1)$$

What is $Perp[(\xi_n, \ldots, \xi_1)]$?
OSS: In case of a uniform distribution $P(\xi_n = w | \ldots) = \frac{1}{|Dict|}$ ...

Cross-entropy

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If we have two distributions (collections of probabilities) $p(x)$ and $q(x)$ on Ω_ξ, then the *cross entropy* of p with respect to q is given by:

$$H_p[q] = -\sum_{x \in \Omega_\xi} p(x) \ln q(x)$$

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Minimality

$$H_p[q] = -\sum_{x \in \Omega_\xi} p(x) \ln q(x) \geq -\sum_{x \in \Omega_\xi} p(x) \ln p(x) \quad \forall q$$

implies that the cross entropy of a distribution q w.r.t. another distribution p is **minimal** when q is identical to p.

Cross-entropy as a Norm

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$$D[p||q] = \sum_{x \in \Omega_\xi} p(x) \ln \frac{p(x)}{q(x)} = H_p[q] - H[p]$$

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$$D[p||q] = 0 \qquad \textbf{iff } q = p$$

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KL distance as a norm?

Unfortunately, as

$$D[p||q] \neq D[q||p]$$

the KL distance is *not* a valid metric in the classical terms. It is a *measure of the dissimilarity* between p and q.

Norm

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What makes a function a norm? Any binary mapping m between a set of objects $D \times D$ and the real numbes is a norm **iff**:

Axioms

- (*Positive*) $m(X,Y) \geq 0 \qquad \forall X, Y \in D$ whereas $m(X,Y) = 0 \rightarrow X = Y$.

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- (*Simmetry*) $m(X,Y) = m(Y,X) \qquad \forall X,Y \in D$

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- (*Triangle inequality*) $m(X,Y) \leq m(X,Z) + m(Z,Y) \qquad \forall X,Y,Z \in D$

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Euclidean Norm

$$\sqrt[2]{\sum_{x \in \Omega(\xi)} (p(x) - q(x))^2}$$

References

Elementary Information Theory

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